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The Power Series Method for Nonlocal and Nonlinear Evolution Equations

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ABSTRACT. *The initial value problem for a 4-parameter family of nonlocal and nonlinear evo- lution equations with data in a space of analytic functions is solved by using a power series method in abstract Banach spaces. In addition to determining the power series expansion of the solution, this method also provides an estimate of the analytic lifespan expressed in terms of the norm of the initial data, thus establishing an abstract Cauchy-Kovalevsky type theorem for these equations.*

I. INTRODUCTION AND RESULTS

In this work we prove an abstract Cauchy-Kovalevsky theorem for the following 4-parameter family of Camassa-Holm type equations

$$u_t + u^k u_x - au^{k-2}u_x^3 + \partial_x (1-\partial_x^2)^{-1} \left[\frac{b}{k+1} u^{k+1} + c\, u^{k-1}u_x^2 - a(k-2)\, u^{k-3}u_x^4\right]$$
$$+ (1-\partial_x^2)^{-1} \left[k(k+2) - 8a - b - c(k+1)\right] u^{k-2}u_x^3 - 3a(k-2)\, u^{k-3}u_x^3 u_{xx} = 0, \qquad (1.1)$$

which was introduced in [HMa1] and is referred there as the *k-abc*-equation. The three parameters a, b and c range over the real numbers while k is a positive integer, whose value depends on a. If $a \neq 0$ then $k \geq 2$ and the presence of the term $au^{k-2}u_x^3$ makes *k-abc*-equation a nonlocal and nonlinear equation which is not quasilinear. For $k = 2$ and $c = (6 - 6a - b)/2$, we obtain the *ab*-family of equations (*ab*-equation) with cubic nonlinearities

$$u_t + u^2u_x - au_x^3 + \partial_x (1-\partial_x^2)^{-1} \left[\frac{b}{3}u^3 + \frac{6-6a-b}{2} uu_x^2\right] + (1-\partial_x^2)^{-1} \frac{2a+b-2}{2} u_x^3 = 0, \quad (1.2)$$

which was also introduced in [HMa1] and which contains two well-known integrable equations with cubic nonlinearities. In fact, for $a = 1/3$ and $b = 2$ the *ab*-equation gives the Fokas-Olver-Rosenau-Qiao (FORQ) equation (also known as the modified Camassa-Holm equation)

$$\partial_t u + u^2\partial_x u - \tfrac{1}{3}(\partial_x u)^3 + \partial_x(1-\partial_x^2)^{-1} \left[\frac{2}{3}u^3 + u(\partial_x u)^2\right] + (1-\partial_x^2)^{-1} \frac{1}{3}(\partial_x u)^3 = 0, \qquad (1.3)$$

which was derived in different ways by Fokas [F], Olver and Rosenau [OR] and Qiao [Q], and also appeared in a work by Fuchssteiner [Fu]. For $a = 0$ and $b = 3$ the *ab*-equation gives the

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Novikov equation (NE)

$$u_t + u^2u_x + \partial_x(1-\partial_x^2)^{-1} \left[u^3 + \frac{3}{2}uu_x^2\right] + (1-\partial_x^2)^{-1} \frac{1}{2}u_x^3 = 0, \qquad (1.4)$$

which was derived by V. Novikov in [N1], where he provides a classification of all integrable CH-type equations with quadratic and cubic nonlinearities.

Finally, for $a = 0$ and $c = (3k - b)/2$ the *k-abc*-equation makes sense for all $k \geq 1$ and gives the following generalized Camassa-Holm equation (g-*kb*CH)

$$u_t + u^k u_x + (1-\partial_x^2)^{-1}\partial_x \left[\frac{b}{k+1} u^{k+1} + \frac{3k-b}{2}u^{k-1}u_x^2\right] + (1-\partial_x^2)^{-1} \frac{(k-1)(b-k)}{2}u^{k-2}u_x^3 = 0, \qquad (1.5)$$

which is a quasilinear equation with $(k+1)$ order nonlinearities and which was studied in [HH2] and [GH]. When $k = 1$ the g-*kb*CH equation gives the well-known *b*-equation

$$\underbrace{m_t}_{\text{evolution}} + \underbrace{um_x}_{\text{convection}} + \underbrace{bu_x m}_{\text{stretching}} = 0, \quad m = u - u_{xx}, \tag{1.6}$$

having quadratic nonlinearities. In this local form it was introduced by Holm and Staley [HS1, HS2] and it expresses a balance between evolution, convection and stretching. The *b*-equation (1.6) contains two integrable members, namely the Camassa-Holm (CH) equation that corresponds to $b = 2$ and the Degasperis-Procesi (DP) equation that corresponds to $b = 3$. Mikhailov and Novikov [MN] proved that there are no other integrable members of the *b*-equation. Furthermore, V. Novikov [N2] recently proved that the only other integrable member of the g-*kb*CH equation (1.5) apart from CH and DP is the NE (1.4). To summarize, thus far it is known that the *k*-*abc*-equation (1.1) contains four integrable equations, CH, DP, FORQ and NE. However, the existence of other integrable members of the *k*-*abc*-equation remains an open question. In fact, integrability theory provides one of the motivations for studying such nonlocal equations.

Another motivation for studying equations like *k*-*abc*-equation is the quest for equations capturing wave breaking and peaking, which goes back to Whitham, who articulates it in his 1974 book [W] (p. 477) as follows: *"Although both breaking and peaking, as well as criteria for the occurrence of each, are without doubt contained in the equations of the exact potential theory, it is intriguing to know what kind of simpler mathematical equation could include all these phenomena."* It is remarkable that the *k*-*abc*-equation has peakon traveling wave solutions for all values of the four parameters k, a, b and c. These, including multipeakons, have been derived in [HMa1]. The peakon solutions in the non-periodic case can be written in the following form

$$u(x, t) = \gamma e^{-|x-(1-a)\gamma^k t|}, \quad \gamma \in \mathbb{R}. \tag{1.7}$$

When $a = 0$ and $c = (3k - b)/2$, which is the case of the generalized Camassa-Holm equation (1.5), these are of the form $u(x, t) = c^{1/k} e^{-|x-ct|}$ and were derived in [GH], together with the corresponding multipeakon on the line and the circle. In the case $k = 1$, which is the *b*-equation, peakon solutions were derived by Holm and Staley [HS1, HS2], who made the important observation that the *b*-equation has peakon (and multipeakon) traveling wave solutions for all values of b. Of course, it is Camassa and Holm [CH] who observed first that the celebrated CH equation has the peakon (weak) solutions $u(x, t) = ce^{-|x-ct|}$.

The initial value problem of the *k*-*abc*-equation in Sobolev spaces was studied in [HMa2]. More precisely, there the following well-posedness result was obtained. If $a, b, c \in \mathbb{R}$ with $a \neq 0$ and $k \in \mathbb{N}$ with $k \geq 2$, then the Cauchy problem for the *k*-*abc*-equation (1.1) with initial data $u(x, 0) = u_0(x) \in H^s$, $x \in \mathbb{R}$ or $\mathbb{T}$, $s > 5/2$ has a unique solution $u \in C([0, T]; H^s)$. Furthermore, the lifespan T satisfies the estimate

$$T \quad \frac{1}{\|u_0\|^k_{H^s}}. \tag{1.8}$$

Also, in [HMa2] continuity properties of the data-to-solution map are investigated. The case $a = 0$ and $c = (6 - 6a - b)/2$ yields the g-*kb*CH equation, whose well-posedness in Sobolev spaces H^s for all $s > 3/2$ was proved in [HH2]. The Cauchy problem for the integrable members CH, DP, FORQ and NE was studied earlier by many authors. For some results on well-posedness, traveling wave solutions, and other analytic properties of these and related equations we refer the reader to the following works and the references therein [BHP2], [BSS], [BC], [CHT], [CK], [CL], [CM], [CS], [DP], [EY], [FF], [HH1], [HK], [HMPZ], [KL], [LO], [LS], [R], [B], [CKSTT], [KPV], [GLOQ], [HHG], [HGH], [HLS], [KT], [L], and [Ti].

In this work we study the initial value problem for *k*-*abc*-equation when the initial data belong to spaces of analytic functions, on both the line and the circle. More precisely, these spaces are defined as follows. For $s \geq 0$ and $\delta > 0$, on the line these spaces are defined by

$$G^{\delta,s}(\mathbb{R}) = \{\phi \in L^2(\mathbb{R}) : \|\phi\|^2_{G^{\delta,s}(\mathbb{R})} \doteq \int_{\mathbb{R}} \langle \xi \rangle^{2s} e^{2\delta|\xi|} |\hat{\phi}(\xi)|^2 d\xi < \infty\}, \tag{1.9}$$

where $\langle \xi \rangle \doteq (1+\xi^2)^{1/2}$. On the circle the corresponding spaces are

$$G^{\delta,s}(\mathrm{T}) = \{\phi \in L^2(\mathrm{T}) : ||\phi||^2_{G^{\delta,s}(\mathrm{T})} \doteq \sum_{k\in Z} \langle k \rangle^{2s} e^{2\delta|k|} |\hat{\phi}(k)|^2 < \infty\}, \tag{1.10}$$

where $\langle k \rangle \doteq (1+k^2)^{1/2}$. Here, when a result holds for both on the line and the circle then we use the notation $||\cdot||_{\delta,s}$ for the norm and $G^{\delta,s}$ for the space in both cases. We observe that a function ϕ in $G^{\delta,s}$ has an analytic extension to a symmetric strip around the real axis with width δ (see Lemma 2). This δ is called the *radius of analyticity* of ϕ. Since in the next Theorem we will assume that the initial data u_0 is in $G^{1,s+2}$ we would like to point out that in the periodic case an analytic function (i.e. an element of $C^{\omega}(\mathrm{T})$) belongs to a $G^{\delta_0,s}(\mathrm{T})$, for some $\delta_0 > 0$ and any $s \geq 0$. More precisely we have the following result, whose proof is easy and will be omitted.

Lemma 1. *If $u_0 \in C^{\omega}(\mathrm{T})$, there exists $\delta_0 > 0$ such that $u_0 \in G^{\delta_0,s}(\mathrm{T})$ for any $s \geq 0$.*

Next, we state our main result, which is motivated by [BHP1] and [BHP2]. For the sake of simplicity we shall assume that our initial data u_0 belong in $G^{1,s+2}$.

Theorem 1. *Let $s > \frac{1}{2}$. If $u_0 \in G^{1,s+2}$ on the circle or the line, then there exists a positive time T, which depends on the initial data u_0 and s, such that for every $\delta \in (0,1)$, the Cauchy problem for the k-abc-equation (1.1) with initial condition $u(x,0) = u_0(x)$ has a unique solution u which is a holomorphic function in $D(0, T(1-\delta))$ valued in $G^{\delta,s+2}$. Furthermore, the analytic lifespan T satisfies the estimate*

$$T \approx \frac{1}{||u_0||^k_{1,s+2}}. \tag{1.11}$$

A more precise statement of estimate (1.11) is provided in Section 4 (see (4.13)). For the Camassa-Holm equation on the circle, a result similar to Theorem 1 but without an analytic lifespan estimate like (1.11) was proved in [HM]. Furthermore, for CH, DP, NE and FORQ Theorem 1 was proved in [BHP2] using a different approach based on a contraction type argument in an appropriate space which is build from a scale of Banach spaces. Here we are using a power series method, which for the quasilinear g-*kb*CH equation was presented in [BHP1]. The novelty of this work is that it provides a comprehensive treatment of a large family of Camassa-Holm type equations whose local part includes *non-quasilinear* terms like $u^{k-2}u_x^3$. Furthermore, it makes a complete presentation of the autonomous Ovsyannikov theorem using the power series method, on which the proof of Theorem 1 is based. We conclude, by mentioning that there are many versions of the abstract Cauchy-Kovalevsky theorem proved in a variety of ways. Many of these works are motivated by water wave models and the Euler equations. For more information about these we refer the reader to Baouendi and Goulaouic [BG1], [BG2], Ovsyannikov [O1], [O2], Treves [Tre1], [Tre2], Nirenberg [Nr], Nishida [Ns], Caflisch [C], Safonov [S], and the references therein.

The paper is organized as follows. In Section 2 we describe the spaces used and prove the needed properties. In Section 3, following the work of Treves, we presents the proof of the autonomous Ovsyannikov theorem which solves an abstract Cauchy problem by the power series method. Finally, in Section 4 we apply the power series method to the *k-abc*-equation and derive the estimates needed for this method.

2. $G^{\delta,s}$ SPACES AND RESULTS

We begin with the properties of the $G^{\delta,s}$ and the estimates needed to prove our main result. The next two lemmas provide a better understanding of the spaces $G^{\delta,s}$ and their properties. One can easily prove these results.

Lemma 2. *Let $\phi \in G^{\delta,s}$. Then, ϕ has an analytic extension to a symmetric strip around the real axis of width δ, for $s \geq 0$ in the periodic case and $s > \frac{1}{2}$ in the non-periodic case.*

Lemma 3. *If $0 < \delta' < \delta \leq 1$, $s \geq 0$ and $\phi \in G^{\delta,s}$ on the circle or the line, then*

$$||\partial_x\phi||_{\delta',s} \le \frac{e^{-1}}{\delta-\delta'}||\phi||_{\delta,s} \tag{2.1}$$
$$||\partial_x\phi||_{\delta,s} \le ||\phi||_{\delta,s+1} \tag{2.2}$$
$$||(1-\partial_x^2)^{-1}\phi||_{\delta,s+2} = ||\phi||_{\delta,s} \tag{2.3}$$
$$||(1-\partial_x^2)^{-1}\phi||_{\delta,s} \le ||\phi||_{\delta,s} \tag{2.4}$$
$$||\partial_x(1-\partial_x^2)^{-1}\phi||_{\delta,s} \le ||\phi||_{\delta,s}. \tag{2.5}$$

Furthermore, we shall need to prove an algebra property for these spaces, which is the main result in the following lemma.

Lemma 4. *For $\phi \in G^{\delta,s}$ on the circle or the line the following properties hold true:*

1) If $0 < \delta' < \delta$ and $s \ge 0$, then $||\cdot||^2_{\delta',s} \le ||\cdot||^2_{\delta,s}$; i.e. $G^{\delta,s} \hookrightarrow G^{\delta',s}$.

2) If $0 < s' < s$ and $\delta > 0$, then $||\cdot||^2_{\delta,s'} \le ||\cdot||^2_{\delta,s}$; i.e. $G^{\delta,s} \hookrightarrow G^{\delta,s'}$.

3) For $s > 1/2$ and $\phi, \psi \in G^{\delta,s}$ we have

$$||\phi\psi||_{\delta,s} \le c_s||\phi||_{\delta,s}||\psi||_{\delta,s}, \tag{2.6}$$

where $c_s = \sqrt{2(1+2^{2s})\sum_{k=0}^{\infty}\frac{1}{\langle k\rangle^{2s}}}$ in the periodic case and $c_s = \sqrt{2(1+2^{2s})\int_0^{\infty}\frac{1}{\langle\xi\rangle^{2s}}d\xi}$ in the non-periodic case.

Proof of Lemma 4. We will provide the proof in the periodic case. The proof in the non-periodic case is similar. Properties (1) and (2) follow directly from the definition of the spaces $G^{\delta,s}$ and the corresponding norms. Therefore, we restrict our attention to the proof of the algebra property (3), which reads as follows

$$||\phi\psi||^2_{\delta,s} = \sum_{k\in Z}\langle k\rangle^{2s}e^{2\delta|k|}|\widehat{\phi\psi}(k)|^2 = ||\langle k\rangle^{s}e^{\delta|k|}\widehat{\phi\psi}(k)||^2_{l^2(Z)}$$
$$= ||\langle k\rangle^{s}e^{\delta|k|}\sum_{n\in Z}\hat\phi(n)\hat\psi(k-n)||^2_{l^2(Z)} \le c_s^2||\phi||^2_{\delta,s}||\psi||^2_{\delta,s}. \tag{2.7}$$

Defining f and g by $\hat f(k) = \langle k\rangle^{s}e^{\delta|k|}\hat\phi(k)$, and $\hat g(k) = \langle k\rangle^{s}e^{\delta|k|}\hat\psi(k)$, we see that the algebra property (2.7) is equivalent to

$$\left\|\langle k\rangle^{s}e^{\delta|k|}\sum_{n\in Z}\frac{e^{-\delta|n|}\hat f(n)}{\langle n\rangle^{s}}\frac{e^{-\delta|k-n|}\hat g(k-n)}{\langle k-n\rangle^{s}}\right\|^2_{l^2(Z)} \le c_s^2||\hat f||^2_{l^2(Z)}||\hat g||^2_{l^2(Z)}. \tag{2.8}$$

Furthermore, using the triangle inequality $|k| \le |n| + |k-n|$ we notice that in order to prove (2.8) it suffices to show that

$$\left\|\langle k\rangle^{s}\sum_{n\in Z}\frac{\hat f(n)}{\langle n\rangle^{s}}\frac{\hat g(k-n)}{\langle k-n\rangle^{s}}\right\|^2_{l^2(Z)} \le c_s^2||\hat f||^2_{l^2(Z)}||\hat g||^2_{l^2(Z)}. \tag{2.9}$$

Using the Cauchy-Schwarz inequality we have

$$\left|\sum_{n\in Z}\frac{\hat f(n)}{\langle n\rangle^{s}}\frac{\hat g(k-n)}{\langle k-n\rangle^{s}}\right|^2 \le \sum_{n\in Z}\frac{1}{\langle n\rangle^{2s}\langle k-n\rangle^{2s}}\sum_{n\in Z}|\hat f(n)|^2|\hat g(k-n)|^2,$$

which gives the following bound for the left-hand side of (2.9)

$$\left\|\langle k\rangle^{s}\sum_{n\in Z}\frac{\hat f(n)}{\langle n\rangle^{s}}\frac{\hat g(k-n)}{\langle k-n\rangle^{s}}\right\|^2_{l^2(Z)} \le \left\|\langle k\rangle^{s}\left(\sum_{n\in Z}\frac{1}{\langle n\rangle^{2s}\langle k-n\rangle^{2s}}\right)^{1/2}\left(\sum_{n\in Z}|\hat f(n)|^2|\hat g(k-n)|^2\right)^{1/2}\right\|^2_{l^2(Z)}$$
$$\le \sup_{k\in Z}\left(\langle k\rangle^{2s}\sum_{n\in Z}\frac{1}{\langle n\rangle^{2s}\langle k-n\rangle^{2s}}\right)\left\|\left(\sum_{n\in Z}|\hat f(n)|^2|\hat g(k-n)|^2\right)^{1/2}\right\|^2_{l^2(Z)}. \tag{2.10}$$

Now, we need the following estimate, whose proof is given after the one of algebra property.

Lemma 5. *For $l > 1/2$, $a \in Z$ and $c_l = \sqrt{2(1+2^{2l})\sum_{k=0}^{\infty}\frac{1}{\langle k\rangle^{2l}}}$ we have*

$$\sum_{k\in Z}\frac{1}{\langle k\rangle^{2l}\langle k-a\rangle^{2l}} \le \frac{c_l^2}{\langle a\rangle^{2l}}. \tag{2.11}$$

In the non-periodic case, for $a \in \mathbb{R}$ *and* $c_l = 2(1+2^{2l})\int_{\xi\geq 0}\frac{1}{\langle \xi\rangle^{2l}}d\xi$ *we have*

$$\int_{\mathbb{R}} \frac{1}{\langle x\rangle^{2l}\langle x-a\rangle^{2l}}dx \leq \frac{c_l^2}{\langle a\rangle^{2l}}. \tag{2.12}$$

Since $s > 1/2$, combining (2.10) and (2.11) and interchanging the order of summations (Fubini's theorem) we get

$$\left\| \langle k\rangle^{s} \sum_{n\in Z} \frac{\hat{f}(n)\hat{g}(k-n)}{\langle n\rangle^{s}\langle k-n\rangle^{s}} \right\|^2_{l^2(Z)} \leq c_s^2 \sum_{n\in Z}\sum_{k\in Z} |\hat{f}(n)|^2|\hat{g}(k-n)|^2 = c_s^2\|\hat{g}\|^2_{l^2(Z)}\sum_{n\in Z}|\hat{f}(n)|^2 = c_s^2\|\hat{f}\|^2_{l^2(Z)}\|\hat{g}\|^2_{l^2(Z)},$$

which completes the proof of Lemma 4. Q

Proof of Lemma 5. If $a = 0$ then it is easy to see that inequality (2.11) holds true. If a is a negative integer then making the change of variables $m = k - a$ the estimate is reduced to the case where a is replaced with $-a$. Thus, it suffices prove inequality (2.11) only when a is a positive integer. For this we shall use the notation $[x]$, which stands for the biggest integer that is less than or equal to x. Thus, $x - 1 < [x] \leq x$. Now, we begin by decomposing our sum as follows

$$S \doteq \sum_{k\in Z} \frac{1}{\langle k\rangle^{2l}\langle k-a\rangle^{2l}} = \sum_{-\infty<k\leq 0} \frac{1}{\langle k\rangle^{2l}\langle k-a\rangle^{2l}} + \sum_{1\leq k\leq [a/2]} \frac{1}{\langle k\rangle^{2l}\langle k-a\rangle^{2l}} + \sum_{[a/2]+1\leq k\leq [a]} \frac{1}{\langle k\rangle^{2l}\langle k-a\rangle^{2l}} + \sum_{[a]+1\leq k<\infty} \frac{1}{\langle k\rangle^{2l}\langle k-a\rangle^{2l}} \doteq S_1 + S_2 + S_3 + S_4.$$

By setting $\vartheta = -k$ we obtain

$$S_1 = \sum_{-\infty<k\leq 0} \frac{1}{(1+k^2)^l(1+(k-a)^2)^l} \leq \frac{1}{\langle a\rangle^{2l}} \sum_{\vartheta=0}^{\infty} \frac{1}{\langle \vartheta\rangle^{2l}}.$$

By noticing that in S_2 we have $1 + (k-a)^2 \geq 1 + a^2/4 = \frac{4+a^2}{4} > \frac{1+a^2}{4}$ we obtain

$$S_2 \leq \frac{2^{2l}}{\langle a\rangle^{2l}} \sum_{k=1}^{\infty} \frac{1}{\langle k\rangle^{2l}}.$$

We also notice that in S_3 we have $1 + k^2 \geq 1 + a^2/4 > \frac{1+a^2}{4}$ and $k - a \leq 0$ and therefore we obtain

$$S_3 \leq \frac{2^{2l}}{\langle a\rangle^{2l}} \sum_{[a/2]+1\leq k\leq 2[a]} \frac{1}{\langle k-a\rangle^{2l}} \leq \frac{2^{2l}}{\langle a\rangle^{2l}} \sum_{k-a\leq 0} \frac{1}{\langle k-a\rangle^{2l}} = \frac{2^{2l}}{\langle a\rangle^{2l}} \sum_{\vartheta=0}^{\infty} \frac{1}{\langle \vartheta\rangle^{2l}}.$$

In S_4 we have $1 + k^2 > 1 + a^2$ and therefore if we set $m = k - a$

$$S_4 \leq \frac{1}{\langle a\rangle^{2l}} \sum_{m=1}^{\infty} \frac{1}{\langle m\rangle^{2l}}.$$

Adding the estimates obtained above for the sums S_1, S_2, S_3 and S_4 gives inequality (2.11). In order to prove (2.12) one has just to replace the sums with integrals. This completes the proof of Lemma 5. Q

3. The Power Series Method

Here, following Treves [Tre1], [Tre2], we prove the autonomous Ovsyannikov theorem using the power series method, which consists of finding a solution for the Cauchy problem

$$\frac{du}{dt} = F(u), \quad u(0) = u_0, \tag{3.1}$$

given by a formal power series

$$u(t) = \sum_{m=0}^{\infty} u_m t^m, \tag{3.2}$$

and estimating the coefficients u_m in order to prove the convergence of the above series. We begin with the following important definition. Let $\{X_\delta\}_{0<\delta\le 1}$ be a decreasing scale of Banach spaces and $u_0 \in X_1$ be given. Also, let $X_0 = \bigcup_{0<\delta<1} X_\delta$.

Definition 1. We say that $F : X_0 \to X_0$ is Ovsyannikov analytic at u_0 if there exist positive constants R, A and C_0 such that for all $k \in Z_+$ and $0 < \delta' < \delta < 1$ we have

$$||D^k F(u)(v_1,\dots, v_k)||_{\delta'} \le \frac{A\, C_0^k k!}{\delta - \delta'} ||v_1||_\delta \dots ||v_k||_\delta \tag{3.3}$$

for all $u \in \{u \in X_\delta : ||u - u_0||_\delta < R\}$ and $(v_1, \dots, v_k) \in X_\delta^k \doteq \underbrace{X_\delta \times \cdots \times X_\delta}_{k}$, where $D^k F$ is the Frechet derivative of F of order k.

Here, we shall prove the following important result.

Theorem 2. *If $u_0 \in X_1$ and F is Ovsyannikov analytic as above, then there exists $T > 0$ such that the Cauchy problem (3.1) has a unique solution which, for every $\delta \in (0, 1)$ is a holomorphic function in $D(0,T(1-\delta))$ valued in X_δ satisfying*

$$\sup_{|t|<T(1-\delta)} \|u(t) - u_0\|_\delta < R, \quad 0 < \delta < 1. \tag{3.4}$$

Moreover, the lifespan T is given by

$$T = \frac{1}{2e^2 A C_0},$$

where the constants R, A and C_0 come from the Definition 1.

From the definition of Ovsyannikov analyticity one can easily prove the following result.

Proposition 1. *If F is Ovsyannikov analytic at $u_0 \in X_1$ then there is $R > 0$ such that, given any pair (δ, δ'), $0 < \delta' < \delta < 1$ and any $u \in B_\delta(u_0; R)$ the Taylor series*

$$\sum_{k=0}^{\infty} \frac{1}{k!} D^k F(u^0) (\underbrace{u - u^0, \dots, u - u^0}_{k})$$

converges absolutely to $F(u)$ in $X_{\delta'}$.

Proof of Theorem 2. Let $0 < \delta' < \delta < 1$. Since $u_0 \in X_1$ and F is Ovsyannikov it follows from Proposition 1 that

$$F(u) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k F(u^0) (\underbrace{u - u^0, \dots, u - u^0}_{k}) \text{ in } X_{\delta'}.$$

Since we want to have $\frac{du}{dt} = F(u)$, we must have

$$u_1 + \sum_{m=1}^{\infty} (m + 1) u_{m+1} t^m$$

$$= F(u_0) + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{\substack{m_1+\cdots+m_k=m \\ m_i\ge 1}} \frac{t^m}{k!} D^k F(u_0)(u_{m_1}, \dots, u_{m_k}).$$

We then conclude that the coefficients of the series are given recursively by

$$u_1 = F(u_0) \tag{3.5}$$

and

$$(m + 1)u_{m+1} = \sum_{k=1}^{\infty} \sum_{\substack{m_1+\cdots+m_k=m \\ m_i\ge 1}} \frac{D^k F(u_0)}{k!} (u_{m_1}, \dots, u_{m_k}), \ m \ge 1. \tag{3.6}$$

We shall use the following lemma.

Lemma 6. *For all $k = 1, 2, \ldots$, we have*

$$\sum_{\substack{m_1+\cdots+m_k=m\\ m_i\ge 1}} \frac{1}{m_1^2\ldots m_k^2} \le \frac{1}{m^2}\left(\frac{2\pi^2}{3}\right)^{k-1}.$$

Proof. We prove the lemma by induction on k. For $k = 1$ the inequality is clear:

$$\sum_{\substack{m_1+\cdots+m_k=m\\ m_i\ge 1}} \frac{1}{m_1^2\ldots m_k^2} = \frac{1}{m^2} = \frac{1}{m^2}\left(\frac{2\pi^2}{3}\right)^{0}.$$

For $k \ge 2$ we can write

$$\sum_{\substack{m_1+\cdots+m_k=m\\ m_i\ge 1}} \frac{1}{m_1^2\ldots m_k^2} = \sum_{l=1}^{m-k+1} \frac{1}{l^2} \sum_{\substack{m_1+\cdots+m_{k-1}=m-l\\ m_i\ge 1}} \frac{1}{m_1^2\ldots m_{k-1}^2}.$$

Hence the induction hypothesis gives

$$\sum_{\substack{m_1+\cdots+m_k=m\\ m_i\ge 1}} \frac{1}{m_1^2\ldots m_k^2} \le \left(\frac{2\pi^2}{3}\right)^{k-2} \sum_{l=1}^{m-1} \frac{1}{l^2(m-l)^2}.$$

We now see that

$$m^2\sum_{l=1}^{m-1}\frac{1}{l^2(m-l)^2} = \sum_{l=1}^{m-k+1}\frac{m^2}{l^2(m-l)^2} = \sum_{l=1}^{m-1}\left(\frac{1}{l}+\frac{1}{m-l}\right)^2$$
$$\le 2\sum_{l=1}^{m-1}\left(\frac{1}{l^2}+\frac{1}{(m-l)^2}\right) \le 4\sum_{l=1}^{\infty}\frac{1}{l^2} = \frac{2\pi^2}{3}.$$

Therefore

$$\sum_{\substack{m_1+\cdots+m_k=m\\ m_i\ge 1}} \frac{1}{m_1^2\ldots m_k^2} \le \left(\frac{2\pi^2}{3}\right)^{k-1},$$

which proves the lemma. Q

We claim that

$$\|u_m\|_\delta \le \frac{\mu}{m^2}\left(\frac{B}{1-\delta}\right)^m, \quad \text{for all } m = 1, 2, \ldots, \tag{3.7}$$

where $\mu = \frac{3}{4C_0\pi^2}$ and $B = 2e^2AC_0$. To prove this fact, we proceed by induction on m. For $m = 1$, we know that $u_1 = F(u_0)$. Then it follows from (3.3) that

$$\|u_1\|_\delta = \|F(u_0)\|_\delta \le \frac{A}{1-\delta} \le \frac{\mu B}{1-\delta}$$

which is true since $\mu B = \frac{3e^2A}{2\pi^2} \ge A$. For $m \ge 2$, we select $\nu = \frac{1-\delta}{m+1} > 0$. We notice that $0 < \delta < \delta + \nu < 1$ and

$$\frac{1}{1-(\delta+\nu)} = \frac{m+1}{m(1-\delta)}. \tag{3.8}$$

Thus, by using (3.6), the induction hypothesis and (3.8) we obtain

$$(m+1)\|u_{m+1}\|_\delta \le \sum_{k=1}^{m}\sum_{m_1+\cdots+m_k=m} AC_0^k\frac{1}{\nu}\|u_{m_1}\|_{\delta+\nu}\ldots\|u_{m_k}\|_{\delta+\nu}$$
$$\le \frac{m+1}{1-\delta}\sum_{k=1}^{m}\sum_{m_1+\cdots+m_k=m}\frac{AC_0^k\mu^k}{m_1^2\ldots m_k^2}\left(\frac{B}{1-\delta}\right)^m\left(1+\frac{1}{m}\right)^m.$$

We then have

$$\|u_{m+1}\|_\delta \le \frac{AC_0e\mu B^m}{m^2(1-\delta)^{m+1}}\sum_{k=1}^{m}\left(\frac{2C_0\mu\pi^2}{3}\right)^{k-1} = \frac{AC_0e\mu B^m}{m^2(1-\delta)^{m+1}}\sum_{k=1}^{m}\left(\frac{1}{2}\right)^{k-1},$$

since $\mu = \frac{3}{4C_0\pi^2}$. Hence,

$$\|u_{m+1}\|_\delta \le \frac{2AC_0e\mu B^m}{m^2(1-\delta)^{m+1}} \le \frac{2AC_0e^2\mu B^m}{(m+1)^2(1-\delta)^{m+1}} = \frac{\mu}{(m+1)^2}\left(\frac{B}{1-\delta}\right)^{m+1},$$

since $B = 2AC_0e^2$. The proof of the claim is complete.

Therefore, for $T = \frac{1}{2e^2AC_0} = B^{-1}$ and for $|t| \le T(1-\delta)$ the series $\sum_{m=0} u_m t^m$ converges absolutely in X_δ for the unique solution to our Cauchy problem. Q

4. THE *k-abc*-EQUATION

In this section we apply the power series method for the *k-abc*-equation. For this we rewrite this equation in the following nonlocal form:

$$\begin{aligned}\partial_t u &= F(u) \doteq -(1-\partial_x^2)^{-1}\partial_x \frac{b+1}{k+1}u^{k+1} + (c-k)u^{k-1}u_x^2 - u^k u_{xx} + 3au^{k-2}u_x^2 u_{xx} \\ &\quad - (1-\partial_x^2)^{-1} [k(k+2) - 9a - b - c(k+1)]u^{k-2}u^3_x - 3a(k-2)u^{k-3}u^3_x u_{xx}\end{aligned} \quad (4.1)$$

We shall prove that there exist positive constants A, C_0 such that

$$||D^j F(u_0)(v_1, \dots, v_j)||_{\delta',s+2} \le \frac{AC^j j!}{\delta - \delta'}||v_1||_{\delta,s+2} \cdots ||v_j||_{\delta,s+2}, \quad (4.2)$$

for all $(v_1,\dots, v_j) \in G^{\delta,s+2}$ and $j = 0, 1, \dots, k+1$ since for $j \ge k+2$ we have

$$||D^j F(u_0)(v_1, \dots, v_j)||_{\delta',s+2} = 0.$$

For simplicity, we shall provide estimate (4.2) only for the term

$$F_1(u) = (1-\partial_x^2)^{-1}\ \ 3a(k-2)u^{k-3}u^3_x u_{xx}\ ,$$

since the other terms can be estimated analogously. By using the following formula for the Frechet derivative of F of order j, $1 \le j \le k+1$, at the point u_0,

$$D^j F(u_0)(v_1,\dots, v_j) = \frac{d}{d\tau_j} \cdots \frac{d}{d\tau_1} F(u_0 + \sum_{i=1}^{j} \tau_i v_i)\Big|_{\tau_1=\dots=\tau_j=0},$$

we obtain, for $v_l \in G^{\delta,s+2}$, $l = 1,\dots, j$,

$$\begin{aligned}&D^j (u^{k-3}u_x^3 u_{xx})\big|_{u_0} (v_1,\dots, v_j) = \\ &= \frac{(k-3)!}{(k-3-j)!} u_0^{k-3-j}(\partial_x u_0)^3(\partial_x^2 u_0) v_1 v_2 \cdots v_j\end{aligned} \quad (4.3)$$

$$+\frac{(k-3)!}{(k-2-j)!}u_0^{k-2-j}(\partial_x u_0)^3\sum_{i=1}^{j}(\partial_x^2 v_i)\tilde{v}_i \tag{4.4}$$

$$+3\frac{(k-3)!}{(k-2-j)!}u_0^{k-2-j}(\partial_x u_0)^2(\partial_x^2 u_0)\sum_{i=1}^{j}(\partial_x v_i)\tilde{v}_i \tag{4.5}$$

$$+3\frac{(k-3)!}{(k-1-j)!}u_0^{k-1-j}(\partial_x u_0)^2\sum_{i=1}^{j}\sum_{\substack{l=1\\ l\neq i}}^{j}(\partial_x^2 v_i)(\partial_x v_l)\tilde{v}_{i,l} \tag{4.6}$$

$$+6\frac{(k-3)!}{(k-1-j)!}u_0^{k-1-j}(\partial_x u_0)(\partial_x^2 u_0)\sum_{i=1}^{j}\sum_{\substack{l=1\\ i<l}}^{j}(\partial_x v_i)(\partial_x v_l)\tilde{v}_{i,l} \tag{4.7}$$

$$+6\frac{(k-3)!}{(k-j)!}u_0^{k-j}(\partial_x u_0)\sum_{i=1}^{j}\sum_{\substack{l=1\\ l\neq i}}^{j}\sum_{\substack{m=1\\ l<m}}^{j}(\partial_x^2 v_i)(\partial_x v_l)(\partial_x v_m)\tilde{v}_{i,l,m} \tag{4.8}$$

$$+6\frac{(k-3)!}{(k-j)!}u_0^{k-j}(\partial_x^2 u_0)\sum_{i=1}^{j}\sum_{\substack{l=1\\ i<l}}^{j}\sum_{\substack{m=1\\ l<m}}^{j}(\partial_x v_i)(\partial_x v_l)(\partial_x v_m)\tilde{v}_{i,l,m} \tag{4.9}$$

$$+6\frac{(k-3)!}{(k+1-j)!}u_0^{k+1-j}\sum_{i=1}^{j}\sum_{\substack{l=1\\ l\neq i}}^{j}\sum_{\substack{m=1\\ l<m}}^{j}\sum_{\substack{n=1\\ m<n}}^{j}(\partial_x^2 v_i)(\partial_x v_l)(\partial_x v_m)(\partial_x v_n)\tilde{v}_{i,l,m,n} \tag{4.10}$$

where we are using the notation $\tilde{v}_{i_1,\dots,i_p}$ to express the product of all the vectors $v_1,\dots,v_j$, except $v_{i_1},\dots,v_{i_p}$, for $1\leq i_1,\dots,i_p\leq j$ distinct from each other and $1\leq p\leq j-1$, and $\tilde{v}_{i_1,\dots,i_j}=1$. Also, the term (4.3) appears only for $j=1,\dots,k-3$, the terms (4.4) and (4.5) appear only for $j=1,\dots,k-2$, the terms (4.6) and (4.7) appear only for $j=2,\dots,k-1$ the terms (4.8) and (4.9) appear only for $j=3,\dots,k$ and finally, the term (4.10) appears only for $j=4,\dots,k+1$.

Without loss of generality, for this term, we will assume that $4\leq j\leq k-3$ since in this case all terms in formulas (4.3) - (4.10) make sense and for $j\in\{0,1,2,3\}$ we can do the computation separately and we obtain an estimate compatible with the case $4\leq j\leq k-3$. By using formula (2.3) in Lemma 3 and triangle inequality, for $v_1,\dots,v_j\in G^{\delta,s+2}$ we have

$$\begin{aligned}
\|D_jF_1(u_0)(v_1,\dots,v_j)\|_{\delta',s+2} \leq 3|a|(k-2)\Bigg[&\frac{(k-3)!}{(k-3-j)!}\|u_0^{k-3-j}(\partial_x u_0)^3(\partial_x^2 u_0)v_1\cdots v_j\|_{\delta',s}\\
&+\frac{(k-3)!}{(k-2-j)!}\|u_0^{k-2-j}(\partial_x u_0)^3\sum_{i=1}^{j}(\partial_x^2 v_i)\tilde{v}_i\|_{\delta',s}\\
&+3\frac{(k-3)!}{(k-2-j)!}\|u_0^{k-2-j}(\partial_x u_0)^2(\partial_x^2 u_0)\sum_{i=1}^{j}(\partial_x v_i)\tilde{v}_i\|_{\delta',s}\\
&+3\frac{(k-3)!}{(k-1-j)!}\|u_0^{k-1-j}(\partial_x u_0)^2\sum_{i=1}^{j}\sum_{\substack{l=1\\ l\neq i}}^{j}(\partial_x^2 v_i)(\partial_x v_l)\tilde{v}_{i,l}\|_{\delta',s}\\
&+6\frac{(k-3)!}{(k-1-j)!}\|u_0^{k-1-j}(\partial_x u_0)(\partial_x^2 u_0)\sum_{i=1}^{j}\sum_{\substack{l=1\\ i<l}}^{j}(\partial_x v_i)(\partial_x v_l)\tilde{v}_{i,l}\|_{\delta',s}\\
&+6\frac{(k-3)!}{(k-j)!}\|u_0^{k-j}(\partial_x u_0)\sum_{i=1}^{j}\sum_{\substack{l=1\\ l\neq i}}^{j}\sum_{\substack{m=1\\ l<m}}^{j}(\partial_x^2 v_i)(\partial_x v_l)(\partial_x v_m)\tilde{v}_{i,l,m}\|_{\delta',s}\\
&+6\frac{(k-3)!}{(k-j)!}\|u_0^{k-j}(\partial_x^2 u_0)\sum_{i=1}^{j}\sum_{\substack{l=1\\ i<l}}^{j}\sum_{\substack{m=1\\ l<m}}^{j}(\partial_x v_i)(\partial_x v_l)(\partial_x v_m)\tilde{v}_{i,l,m}\|_{\delta',s}\\
&+6\frac{(k-3)!}{(k+1-j)!}\|u_0^{k+1-j}\sum_{i=1}^{j}\sum_{\substack{l=1\\ l\neq i}}^{j}\sum_{\substack{m=1\\ l<m}}^{j}\sum_{\substack{n=1\\ m<n}}^{j}(\partial_x^2 v_i)(\partial_x v_l)(\partial_x v_m)(\partial_x v_n)\tilde{v}_{i,l,m,n}\|_{\delta',s}\Bigg].
\end{aligned}$$

Now, from the algebra property in Lemma 4 and (2.2) in Lemma 3, we obtain

$$
\begin{aligned}
&\|D^jF_1(u_0)(v_1,\dots,v_j)\|_{\delta',s+2} \le \\
&\le 3|a|c_s^k(k-2)\frac{(k-3)!}{(k-3-j)!}\|u_0\|_{\delta',s}^{k-3-j}\|\partial_x u_0\|_{\delta',s}^3\|\partial_x^2 u_0\|_{\delta',s}\|v_1\|_{\delta',s}\cdots\|v_j\|_{\delta',s} \\
&+\frac{(k-3)!j}{(k-2-j)!}\|u_0\|_{\delta',s}^{k-2-j}\|\partial_x u_0\|_{\delta',s}^3\|v_1\|_{\delta',s+2}\cdots\|v_j\|_{\delta',s+2} \\
&+3\frac{(k-3)!j}{(k-2-j)!}\|u_0\|_{\delta',s}^{k-2-j}\|\partial_x u_0\|_{\delta',s}^2\|\partial_x^2 u_0\|_{\delta',s}\|v_1\|_{\delta',s+1}\cdots\|v_j\|_{\delta',s+1} \\
&+3\frac{(k-3)!j(j-1)}{(k-1-j)!}\|u_0\|_{\delta',s}^{k-1-j}\|\partial_x u_0\|_{\delta',s}^2\|v_1\|_{\delta',s+2}\cdots\|v_j\|_{\delta',s+2} \\
&+6\frac{(k-3)!j(j-1)}{(k-1-j)!}\|u_0\|_{\delta',s}^{k-1-j}\|\partial_x u_0\|_{\delta',s}\|\partial_x^2 u_0\|_{\delta',s}\|v_1\|_{\delta',s+1}\cdots\|v_j\|_{\delta',s+1} \\
&+6\frac{(k-3)!j(j-1)(j-2)}{(k-j)!}\|u_0\|_{\delta',s}^{k-j}\|\partial_x u_0\|_{\delta',s}\|v_1\|_{\delta',s+2}\cdots\|v_j\|_{\delta',s+2} \\
&+6\frac{(k-3)!j(j-1)(j-2)}{(k-j)!}\|u_0\|_{\delta',s}^{k-j}\|\partial_x^2 u_0\|_{\delta',s}\|v_1\|_{\delta',s+1}\cdots\|v_j\|_{\delta',s+1} \\
&+\frac{6e^{-1}}{\delta-\delta'}\frac{(k-3)!j(j-1)(j-2)(j-3)}{(k+1-j)!}\|u_0\|_{\delta',s}^{k+1-j}\|v_1\|_{\delta',s+1}\cdots\|v_j\|_{\delta',s+1}.
\end{aligned}
$$

Finally, by using lemmas 3 and 4, for $0<\delta'<\delta\le 1$ and $v_1,\dots,v_j\in G^{\delta,s+2}$, with $s>1/2$, we can estimate

$$
\begin{aligned}
&\|D^jF_1(u_0)(v_1,\dots,v_j)\|_{\delta',s+2} \\
&\le \frac{3|a|c_s^k(k-2)e^{-1}}{\delta-\delta'}\|u_0\|_{1,s+2}^{k-j+1}\|v_1\|_{\delta,s+2}\cdots\|v_j\|_{\delta,s+2}\left[\frac{(k-3)!}{(k-3-j)!}+\right. \\
&\left.+\frac{4(k-3)!j!}{(j-1)!(k-2-j)!}+\frac{9(k-3)!j!}{(j-2)!(k-1-j)!}+\frac{12(k-3)!j!}{(j-3)!(k-j)!}+\frac{6(k-3)!j!}{(j-4)!(k+1-j)!}\right].
\end{aligned}
$$

By using the fact that $(k-2)\le 2^{k-3}$ for $k\in\{3,4,\dots\}$ we obtain

$$
\begin{aligned}
&\|D^jF_1(u_0)(v_1,\dots,v_j)\|_{\delta',s+2} \\
&\le \frac{36|a|c_s^k2^{k-3}e^{-1}j!}{\delta-\delta'}\|u_0\|_{1,s+2}^{k-j+1}\|v_1\|_{\delta,s+2}\cdots\|v_j\|_{\delta,s+2}\left[\binom{k-3}{j}+\binom{k-3}{j-1}\right. \\
&\left.+\binom{k-3}{j-2}+\binom{k-3}{j-3}+\binom{k-3}{j-4}\right].
\end{aligned}
$$

Since

$$\binom{k-3}{j}+\binom{k-3}{j-1}+\binom{k-3}{j-2}+\binom{k-3}{j-3}+\binom{k-3}{j-4}\le 2^{k-3},$$

if we take $C_0=\dfrac{1}{\|u_0\|_{1,s+2}}$ and $A_1=|a|c_s^ke^{-1}2^{2k}\|u_0\|_{1,s+2}^{k+1}$ then we have that

$$\|D^jF_1(u_0)(v_1,\dots,v_j)\|_{\delta',s+2}\le\frac{A_1C_0^jj!}{\delta-\delta'}\|v_1\|_{\delta,s+2}\|v_2\|_{\delta,s+2}\cdots\|v_j\|_{\delta,s+2}. \tag{4.11}$$

We can now do similar computations for the other terms of $F(u)$ and get

$$\|D^jF(u_0)(v_1,\dots,v_j)\|_{\delta',s+2}\le\frac{AC_0^jj!}{\delta-\delta'}\|v_1\|_{\delta,s+2}\|v_2\|_{\delta,s+2}\cdots\|v_j\|_{\delta,s+2}, \tag{4.12}$$

where $A=(2|a|+|b+1|+2|c|+|9a+b|+3)c_s^k2^{3k+2}e^{-1}\|u_0\|_{1,s+2}^{k+1}$

Therefore, by Theorem 2 we conclude that the Cauchy problem for the k-abc-equation (1.1) with initial condition $u(x,0)=u_0(x)$ has a unique solution, which for $0<\delta<1$ is a holomorphic function in the disc $D(0,T(1-\delta))$ valued in $G^{\delta,s+2}$. Moreover, the lifespan T is given by

$$T=\frac{1}{2e^2AC_0}=\frac{1}{c\|u_0\|_{1,s+2}^k},\quad \text{where } c=e(2|a|+|b+1|+2|c|+|9a+b|+3)c_s^k2^{3k+3}. \tag{4.13}$$

The proof of Theorem 1 is now complete. Q

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